

# An isoperimetric inequality for extremal Sobolev functions

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## Abstract

Let  $D \subset \mathbf{R}^n$  be a bounded domain with a Lipschitz boundary, let  $1 < p < \frac{2n}{n-2}$ , and let  $\phi$  minimize the ratio  $\|\nabla u\|_{L^2}/\|u\|_{L^p}$ . We prove a reverse-Hölder inequality, finding a lower bound for  $\|\phi\|_{L^{p-1}}$  in terms of  $\|\phi\|_{L^p}$ , in which equality holds if and only if  $D$  is a ball. This result generalizes an inequality due to Payne and Rayner [6, 7] regarding eigenfunctions of the Laplacian.

## 1 Introduction and statement of results

Let  $D \subset \mathbf{R}^n$  be a bounded domain with Lipschitz boundary, and let  $1 < p < \frac{2n}{n-2}$  (or,  $p > 1$  if  $n = 2$ ). For this range of exponents, the Sobolev embedding  $W_0^{1,2}(D) \hookrightarrow L^p(D)$  is compact, and so the infimum

$$C_p(D) = \inf \left\{ \frac{\int_D |\nabla u|^2 d\mu}{(\int_D |u|^p d\mu)^{2/p}} : u \in W_0^{1,2}(D), u \not\equiv 0 \right\} \quad (1.1)$$

is finite and achieved by a nontrivial function  $\phi = \phi_p$ .

We take this opportunity to set notation for the remainder of the paper. We denote the volume element of the usual Lebesgue measure in  $\mathbf{R}^n$  by  $d\mu$ ; when it will be necessary, we will denote the induced area element on a hypersurface  $\Sigma \subset \mathbf{R}^n$  by  $d\sigma$ . We write the appropriate dimensional volume of a set as  $|\Omega|$ , *i.e.* if  $\Omega \subset \mathbf{R}^n$  is an open set then  $|\Omega| = \mu(\Omega)$  and if  $\Sigma \subset \mathbf{R}^n$  is a hypersurface then  $|\Sigma| = \sigma(\Sigma)$ . If  $\mathbf{B}_1 \subset \mathbf{R}^n$  is the unit ball, we denote  $|\mathbf{B}_1| = \omega_n$ , so that  $|\mathbf{B}_r| = \omega_n r^n$  and  $|\partial \mathbf{B}_r| = n\omega_n r^{n-1}$ . The Sobolev space  $W_0^{1,2}(D)$  is the closure of  $C_0^\infty(D)$  under the norm  $\|u\|_{W^{1,2}}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$ .

An extremal function  $\phi$  for (1.1) will solve the boundary value problem in  $D$ :

$$\Delta \phi + \lambda \phi^{p-1} = 0, \quad \phi|_{\partial D} = 0. \quad (1.2)$$

Without loss of generality we can take  $\phi > 0$  inside  $D$ . General regularity results imply that  $\phi \in C_0^\infty(D)$ , and a short integration by parts argument reveals that

$$\lambda = C_p(D) \left( \int_D \phi^p d\mu \right)^{\frac{2-p}{p}}. \quad (1.3)$$

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This sharp Sobolev constant  $\mathcal{C}_p(D)$  and its associated extremal function  $\phi_p$  are both the subject of a vast literature, and incorporate much information relating the function theory and the geometry of  $D$ . In particular, a long string of results (for example, [6, 4, 1, 3]) have uncovered isoperimetric-type inequalities of varying sorts. Our main theorem generalizes the reverse-Hölder inequalities of [6, 7, 2], and has the following form.

**Theorem 1.** *Let  $D \subset \mathbf{R}^n$  be a bounded domain with Lipschitz boundary, let  $\mathcal{C}_p(D)$  be the sharp Sobolev constant defined by (1.1), and let  $\phi$  be its associated extremal function. Let  $D^*$  be a ball with the same volume as  $D$ . Then*

$$\left( \int_D \phi^{p-1} d\mu \right)^2 \geq |D|^{\frac{n-2}{n}} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \left[ \frac{2n^2 \omega_n^{2/n}}{p \mathcal{C}_p(D)} - (n-2) \frac{n \omega_n^{\frac{2}{n} + \frac{p^2-p+2}{p(p-1)}}}{\mathcal{C}_p(D^*)} \right]. \quad (1.4)$$

*Equality holds if and only if  $D$  is a ball.*

**Remark 1.** • *The Hölder inequality implies that for any  $u \in W_0^{1,2}$  we have*

$$\int_D |u|^{p-1} d\mu \leq |D|^{1/p} \left( \int_D |u|^p d\mu \right)^{\frac{p-1}{p}}.$$

*For this reason, upper bounds of the form (1.4) are called reverse-Hölder inequalities.*

- *Observe that we recover the main inequality of [7] in the case  $p = 2$ , and we recover the reverse-Hölder inequality of [2] in the case  $n = 2$ .*

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## 2 Proof of the main theorem

We begin by briefly outlining our strategy for proving (1.4), which we adapted from Payne and Rayner’s proof in [7]. Let  $M = \sup_{x \in D} \phi(x)$ , and for  $0 \leq t \leq M$  we define

$$D_t = \{x \in D : \phi(x) > t\}, \quad \Sigma_t = \{x \in D : \phi(x) = t\}.$$

By Sard’s theorem, we have  $\Sigma_t = \partial D_t$  for almost every value of  $t$ . To prove (1.4) we define the auxilliary function

$$H(t) = \int_{D_t} \phi^{p-1} d\mu = \int_t^M \tau^{p-1} \int_{\Sigma_\tau} \frac{d\sigma}{|\nabla \phi|} d\tau, \quad t \in [0, M].$$

In Section 2.1 we derive lower bounds for the second derivative of  $H$ , and in Section 2.2 we integrate these to obtain several integral inequalities for  $H$  and for powers of  $\phi$ . In Section 2.3 we examine the one-dimensional eigenvalue problem which arises from the radially symmetric case, and in Section 2.4 complete the proof of (1.4).

## 2.1 Differential inequalities

We let  $V(t) = |D_t|$ . Then, by the co-area formula,

$$V'(t) = - \int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|} < 0.$$

Thus  $V$  is a monotone function of  $t$ , and we can invert it to obtain  $t = t(V)$ , with

$$\frac{dt}{dV} = \frac{1}{V'(t)} = - \frac{1}{\int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|}}.$$

This in turn implies that

$$\frac{dH}{dV} = \frac{dH}{dt} \frac{dt}{dV} = \left( -t^{p-1} \int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|} \right) \cdot \left( - \frac{1}{\int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|}} \right) = t^{p-1}, \quad (2.1)$$

a relation which will prove quite useful in our computations. Taking one more derivative shows that

$$\frac{d^2 H}{dV^2} = \frac{d}{dV} (t^{p-1}) = - \frac{(p-1) t^{p-2}}{\int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|}}.$$

**Lemma 1.** *The function  $H$  satisfies*

$$\frac{d^2 H}{dV^2} \geq -(p-1)(t(V))^{p-2} \frac{\Lambda H(V)}{n^2 \omega_n^{2/n} V^{\frac{2(n-1)}{n}}}, \quad V \in [0, |D|]. \quad (2.2)$$

with the boundary conditions  $H(0) = 0$  and  $H'(|D|) = 0$ . Moreover, equality in (2.2) forces  $D$  to be a ball, and forces the function  $\phi$  to be radially symmetric.

*Proof.* By the Cauchy-Schwarz inequality,

$$|\Sigma_t|^2 \leq \left( \int_{\Sigma_t} |\nabla\phi| d\sigma \right) \left( \int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|} \right),$$

which we can rearrange to read

$$\int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|} \geq \frac{|\Sigma_t|^2}{\int_{\Sigma_t} |\nabla\phi| d\sigma}. \quad (2.3)$$

Since  $\Sigma_t$  is a level-set of  $\phi$ , we may use the divergence theorem and (1.2) to obtain

$$\begin{aligned} \int_{\Sigma_t} |\nabla\phi| d\sigma &= - \int_{\Sigma_t} \frac{\partial\phi}{\partial\eta} d\sigma = - \int_{D_t} \Delta\phi d\mu \\ &= \lambda \int_{D_t} \phi^{p-1} d\mu = \lambda H(t). \end{aligned}$$

Combining this with (2.3) we obtain

$$\int_{\Sigma_t} \frac{d\sigma}{|\nabla\phi|} \geq \frac{|\Sigma_t|^2}{\lambda H(t)}. \quad (2.4)$$

By the classical isoperimetric inequality,

$$|\Sigma_t|^2 \geq n^2 \omega_n^{2/n} |D_t|^{\frac{2(n-1)}{n}} = n^2 \omega_n^{2/n} V^{\frac{2(n-1)}{n}}. \quad (2.5)$$

Together with (2.4), this shows that

$$\begin{aligned} \frac{d^2 H}{dV^2} &= -(p-1) \frac{t^{p-2}}{\int_{\Sigma_t} \frac{d\sigma}{|\nabla \phi|}} \geq -(p-1) t^{p-2} \frac{\lambda H(V)}{|\Sigma_t|^2} \\ &\geq -(p-1) t^{p-2} \frac{\lambda H(V)}{n^2 \omega_n^{2/n} V^{\frac{2(n-1)}{n}}}. \end{aligned}$$

Notice that the boundary conditions for this differential inequality are

$$H(0) = 0, \quad H'(|D|) = t^{p-1}|_{t=0} = 0. \quad (2.6)$$

Moreover, we only have equality in (2.2) for each  $V$  in  $[0, |D|]$  if we have equality in (2.5) for almost every  $t$ , which in turn implies that  $\Sigma_t$  is a round sphere for almost every  $t \in [0, M]$ . This is possible only if  $D$  is itself a ball. Also, equality in (2.2) forces equality in (2.3), which implies  $\nabla \phi$  must be constant on each sphere  $\Sigma_t$ , and so  $\phi$  must be radial.  $\square$

We change variables by letting  $\rho = (V/\omega_n)^{1/n}$  be the volume radius of  $D_t$ , so that  $V = |D_t| = \omega_n \rho^n$ . We also define  $\rho_M = (|D|/\omega_n)^{1/n}$ . As a function of  $\rho$ , the function  $H$  satisfies the boundary conditions

$$H(0) = H'(0) = \dots = H^{(n-1)}(0) = 0, \quad H'(\rho_M) = 0. \quad (2.7)$$

**Lemma 2.**

$$\frac{d}{d\rho} \left[ \rho^{1-n} \left( \frac{dH}{d\rho} \right)^{\frac{1}{p-1}} \right] \geq - \frac{\lambda}{(n\omega_n)^{\frac{p-2}{p-1}}} \rho^{1-n} H(\rho), \quad 0 < \rho < \rho_M. \quad (2.8)$$

*Proof.* Taking derivatives, we see that

$$\frac{dH}{dV} = \frac{\rho^{1-n}}{n\omega_n} \frac{dH}{d\rho}, \quad \frac{d^2 H}{dV^2} = \frac{\rho^{1-n}}{n\omega_n} \frac{d}{d\rho} \left( \frac{\rho^{1-n}}{n\omega_n} \frac{dH}{d\rho} \right). \quad (2.9)$$

Substituting these expressions in (2.2) gives

$$\frac{d}{d\rho} \left( \rho^{1-n} \frac{dH}{d\rho} \right) \geq -(p-1) \lambda t^{p-2} \rho^{1-n} H(\rho). \quad (2.10)$$

However,

$$t^{p-1} = \frac{dH}{dV} = \frac{\rho^{1-n}}{n\omega_n} \frac{dH}{d\rho},$$

so that (2.10) becomes

$$\frac{d}{d\rho} \left( \rho^{1-n} \frac{dH}{d\rho} \right) \geq - \frac{(p-1) \lambda}{(n\omega_n)^{\frac{p-2}{p-1}}} \left( \rho^{1-n} \frac{dH}{d\rho} \right)^{\frac{p-2}{p-1}} \rho^{1-n} H(\rho).$$

This we can rewrite as

$$\frac{1}{p-1} \frac{\frac{d}{d\rho} \left( \rho^{1-n} \frac{dH}{d\rho} \right)}{\left( \rho^{1-n} \frac{dH}{d\rho} \right)^{\frac{p-2}{p-1}}} = \frac{d}{d\rho} \left[ \rho^{1-n} \left( \frac{dH}{d\rho} \right)^{\frac{1}{p-1}} \right] \geq -\frac{\lambda}{(n\omega_n)^{\frac{p-2}{p-1}}} \rho^{1-n} H(\rho). \quad \square$$

**Remark 2.** Since (2.8) is really the same as (2.2) rewritten in different variables, equality holds in (2.8) for  $0 < \rho < \rho_M$  if and only if  $D$  is a ball and  $\phi$  is radial.

## 2.2 Integral inequalities

In this section we integrate (2.2) and (2.8) to obtain inequalities for the integral of  $H$  and the integral of powers of  $\phi$ . As each of these inequalities is an integrated form of (2.2) and (2.8), equality holds if and only if  $D$  is a ball and  $\phi$  is radial.

**Lemma 3.**

$$\begin{aligned} \left( \int_D \phi^{p-1} d\mu \right)^2 &\geq \frac{2n^2 \omega_n^{2/n} |D|^{\frac{n-2}{n}}}{p C_p(D)} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \\ &\quad - \frac{n-2}{n} |D|^{\frac{n-2}{n}} \int_0^{|D|} V^{\frac{2(1-n)}{n}} H^2(V) dV. \end{aligned} \quad (2.11)$$

*Proof.* We multiply the inequality (2.2) by  $\frac{p}{p-1} V \left( \frac{dH}{dV} \right)^{1/(p-1)}$  and integrate from 0 to  $|D|$ . Upon integration, the left hand side becomes

$$\begin{aligned} \int_0^{|D|} \frac{p}{p-1} V \left( \frac{dH}{dV} \right)^{1/(p-1)} \frac{d^2 H}{dV^2} dV &= \int_0^{|D|} V \frac{d}{dV} \left[ \left( \frac{dH}{dV} \right)^{p/(p-1)} \right] dV \\ &= V \left( \frac{dH}{dV} \right)^{p/(p-1)} \Big|_0^{|D|} - \int_0^{|D|} \left( \frac{dH}{dV} \right)^{p/(p-1)} dV \\ &= - \int_0^{|D|} (t^{p-1}(V))^{p/(p-1)} dV \\ &= - \int_0^{|D|} t^p(V) dV = - \int_D \phi^p d\mu. \end{aligned}$$

The boundary terms in the integration by parts vanished since  $H'(|D|) = 0$ , while (2.1) was used at the third step. On the other hand, using (2.1) again, the right hand side becomes

$$\begin{aligned} -\frac{p\lambda}{n^2 \omega_n^{n/2}} \int_0^{|D|} V \left( \frac{dH}{dV} \right)^{1/(p-1)} t^{p-2} H(t) V^{\frac{2(1-n)}{n}} dV \\ &= -\frac{p\lambda}{n^2 \omega_n^{n/2}} \int_0^{|D|} t^{p-2} \left( \frac{dH}{dV} \right)^{\frac{2-p}{p-1}} V^{\frac{2-n}{n}} H(V) \frac{dH}{dV} dV \\ &= -\frac{p\lambda}{n^2 \omega_n^{n/2}} \int_0^{|D|} t^{p-2} (t^{p-1})^{\frac{2-p}{p-1}} V^{\frac{2-n}{n}} H(V) \frac{dH}{dV} dV \\ &= -\frac{p\lambda}{n^2 \omega_n^{n/2}} \int_0^{|D|} V^{\frac{2-n}{n}} H(V) \frac{dH}{dV} dV \end{aligned}$$

We combine these last two equations and replace  $\lambda$  by  $\mathcal{C}_p(D) \left( \int_D \phi^p d\mu \right)^{(2-p)/p}$  to obtain

$$\begin{aligned}
\left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} &\leq \frac{p \mathcal{C}_p(D)}{n^2 \omega_n^{n/2}} \int_0^{|D|} V^{\frac{2-n}{n}} H(V) \frac{dH}{dV} dV \\
&= \frac{p \mathcal{C}_p(D)}{2n^2 \omega_n^{2/n}} \int_0^{|D|} V^{\frac{2-n}{n}} \frac{d}{dV} (H^2(V)) dV \\
&= \frac{p \mathcal{C}_p(D)}{2n^2 \omega_n^{2/n}} \left[ |D|^{\frac{2-n}{n}} \left( \int_D \phi^{p-1} d\mu \right)^2 + \frac{n-2}{n} \int_0^{|D|} H^2(V) V^{\frac{2(1-n)}{n}} dV \right],
\end{aligned}$$

which we can rearrange to give (2.11).  $\square$

**Lemma 4.**

$$\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{dH}{d\rho} \right)^{\frac{p}{p-1}} d\rho \leq \frac{\lambda}{(n\omega_n)^{\frac{p-2}{p-1}}} \int_0^{\rho_M} \rho^{1-n} H^2(\rho) d\rho. \quad (2.12)$$

*Equality holds if and only if  $D$  is a ball.*

*Proof.* We multiply (2.8) by  $H$  and integrate from 0 to  $\rho_M$ . The boundary conditions (2.7) imply that  $\rho^{1-n} \frac{dH}{d\rho}$  is bounded at 0. Hence the boundary terms vanish in the integration parts below, and we obtain that

$$\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{dH}{d\rho} \right)^{\frac{p}{p-1}} d\rho = \int_0^{\rho_M} \left[ \rho^{1-n} \frac{dH}{d\rho} \right]^{\frac{1}{p-1}} \frac{dH}{d\rho} d\rho \leq \frac{\lambda}{(n\omega_n)^{\frac{p-2}{p-1}}} \int_0^{\rho_M} \rho^{1-n} H^2(\rho) d\rho.$$

$\square$

**Lemma 5.** *With  $\phi$ ,  $H$ , and  $\rho$  defined as above,*

$$\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{dH}{d\rho} \right)^{\frac{p}{p-1}} d\rho = (n\omega_n)^{\frac{1}{p-1}} \int_D \phi^p d\mu. \quad (2.13)$$

*Proof.* We use (2.1) and (2.9) to conclude

$$\begin{aligned}
\int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{dH}{d\rho} \right)^{\frac{p}{p-1}} d\rho &= \int_0^{\rho_M} (\rho^{1-n})^{\frac{p}{p-1}} \left( \frac{dH}{d\rho} \right)^{\frac{p}{p-1}} \rho^{n-1} d\rho \\
&= \int_0^{|D|} \left( \rho^{1-n} \frac{dH}{d\rho} \right)^{\frac{p}{p-1}} \frac{dV}{n\omega_n} \\
&= \int_0^{|D|} \left( n\omega_n \frac{dH}{dV} \right)^{\frac{p}{p-1}} \frac{dV}{n\omega_n} \\
&= (n\omega_n)^{\frac{1}{p-1}} \int_0^{|D|} \left( \frac{dH}{dV} \right)^{\frac{p}{p-1}} dV \\
&= (n\omega_n)^{\frac{1}{p-1}} \int_0^{|D|} (t^{p-1})^{\frac{p}{p-1}} dV \\
&= (n\omega_n)^{\frac{1}{p-1}} \int_0^{|D|} t^p dV = (n\omega_n)^{\frac{1}{p-1}} \int_D \phi^p d\mu. \quad \square
\end{aligned}$$

**Corollary 2.**

$$n\omega_n \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \leq \mathcal{C}_p(D) \int_0^{\rho_M} \rho^{1-n} H^2(\rho) d\rho. \quad (2.14)$$

Moreover, we have equality if and only if  $D$  is a ball and  $\phi$  is radial.

*Proof.* Combine (2.12), (2.13), and (1.3).  $\square$

**Lemma 6.**

$$\begin{aligned} \left( \int_D \phi^{p-1} d\mu \right)^2 &\geq \frac{2n^2 \omega_n^{2/n} |D|^{\frac{n-2}{n}}}{p \mathcal{C}_p(D)} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \\ &\quad - (n-2) \omega_n^{\frac{2-n}{n}} |D|^{\frac{n-2}{n}} \int_0^{\rho_M} \rho^{1-n} H^2(\rho) d\rho. \end{aligned} \quad (2.15)$$

Equality holds if and only if  $D$  is a ball.

*Proof.* Since  $\rho(V) = (V/\omega_n)^{1/n}$ , we have

$$n\omega_n^{1/n} \frac{d\rho}{dV} V^{\frac{n-1}{n}} = 1,$$

so that

$$\begin{aligned} \int_0^{|D|} V^{\frac{2(1-n)}{n}} H^2(V) dV &= \int_0^{|D|} H^2(\rho) \omega_n^{\frac{2(1-n)}{n}} \rho^{2(1-n)} n\omega_n^{1/n} (\omega_n \rho^n)^{\frac{n-1}{n}} \frac{d\rho}{dV} dV \\ &= n\omega_n^{\frac{2-n}{n}} \int_0^{\rho_M} \rho^{1-n} H^2(\rho) d\rho. \end{aligned}$$

Putting this into (2.11) gives (2.15).  $\square$

### 2.3 The radially symmetric case

Motivated by (2.12) and (2.7), we define  $\Lambda_*$  by

$$\Lambda_* = \inf \left\{ \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \right)^{\frac{2(p-1)}{p}} \middle/ \int_0^{\rho_M} \rho^{1-n} f^2(\rho) d\rho \right\} \quad (2.16)$$

where the infimum is over all functions on  $[0, \rho_M]$  for which

$$f(0) = f'(0) = \dots = f^{(n-1)}(0) = 0 = f'(\rho_M), \quad f \not\equiv 0. \quad (2.17)$$

**Remark 3.** Notice that we have rescaled the numerator to make the quotient scale-invariant. This does not, however, affect the Euler-Lagrange equation involved.

**Lemma 7.** The Euler-Lagrange equation for the variational problem (2.16), with the boundary conditions (2.17), is

$$f''(\rho) - \frac{n-1}{\rho} f'(\rho) + \Lambda [\rho^{1-n} f'(\rho)]^{\frac{p-2}{p-1}} f(\rho) = 0. \quad (2.18)$$

*Proof.* Since the ratio defining  $\Lambda_*$  is scale-invariant, we may either restrict our attention to either of the constrained critical point problems:

$$\text{minimize } \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \text{ subject to } \int_0^{\rho_M} \rho^{1-n} f^2 d\rho = \text{constant}$$

or

$$\text{maximize } \int_0^{\rho_M} \rho^{1-n} f^2 d\rho \text{ subject to } \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho = \text{constant}.$$

Regardless, the method of Lagrange multipliers implies that a constrained critical point  $f$  satisfies

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{df}{d\rho} + \epsilon \frac{dg}{d\rho} \right)^{\frac{p}{p-1}} d\rho = \Lambda \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_0^{\rho_M} \rho^{1-n} [f(\rho) + \epsilon g(\rho)]^2 d\rho,$$

for any admissible  $g$ . Having evaluated these derivatives, we use the boundary conditions (2.17) to see that  $\rho^{1-n} f'(\rho)$  is bounded at 0 and that consequently the boundary terms arising from integration by parts vanish, and see that

$$\begin{aligned} 2\lambda \int_0^{\rho_M} \rho^{1-n} f(\rho) g(\rho) d\rho &= \frac{p}{p-1} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{df}{d\rho} \right)^{\frac{1}{p-1}} \frac{dg}{d\rho} d\rho \\ &= -\frac{p}{p-1} \int_0^{\rho_M} g(\rho) \frac{d}{d\rho} \left[ \rho^{\frac{1-n}{p-1}} \left( \frac{df}{d\rho} \right)^{\frac{1}{p-1}} \right] d\rho \\ &= -\frac{p}{p-1} \int_0^{\rho_M} g(\rho) \left[ \frac{1}{p-1} \rho^{\frac{1-n}{p-1}} \left( \frac{df}{d\rho} \right)^{\frac{2-p}{p-1}} \frac{d^2 f}{d\rho^2} + \frac{1-n}{p-1} \rho^{\frac{2-p-n}{p-1}} \left( \frac{df}{d\rho} \right)^{\frac{1}{p-1}} \right] d\rho. \end{aligned}$$

This must hold for all choices of  $g$ , hence (absorbing a factor of  $2p/(p-1)^2$  into the Lagrange multiplier  $\Lambda$ ) we must have

$$\begin{aligned} 0 &= \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{2-p}{p-1}} f''(\rho) - (n-1) \rho^{\frac{2-p-n}{p-1}} f'(\rho)^{\frac{1}{p-1}} + \Lambda \rho^{1-n} f(\rho) \\ &= \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{2-p}{p-1}} \left[ f''(\rho) - (n-1) \rho^{-1} f'(\rho) + \Lambda [\rho^{1-n} f'(\rho)]^{\frac{p-2}{p-1}} f(\rho) \right], \end{aligned}$$

as claimed.  $\square$

**Lemma 8.** *Let  $D^*$  be the ball  $\mathbf{B}_{\rho_M}$  of radius  $\rho_M$ . Then,*

$$\Lambda_* \leq (n\omega_n)^{\frac{2-p}{p}} \mathcal{C}_p(D^*). \quad (2.19)$$

*Proof.* We use the function  $H(\rho)$  for the ball  $\mathbf{B}_{\rho_M}$  as a test function for the quotient



defining  $\Lambda_*$  and use the inequalities (2.12), (1.3), and (2.13):

$$\begin{aligned}
\Lambda_* &\leq \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} H'(\rho)^{\frac{p}{p-1}} d\rho \right)^{\frac{2(p-1)}{p}} \bigg/ \int_0^{\rho_M} \rho^{1-n} H^2(\rho) d\rho \\
&\leq \frac{\Lambda}{(n\omega_n)^{\frac{p-2}{p-1}}} \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} H'(\rho)^{\frac{p}{p-1}} d\rho \right)^{\frac{p-2}{p}} \\
&= \frac{1}{(n\omega_n)^{\frac{p-2}{p-1}}} \mathcal{C}_p(D^*) \left( \int_{D^*} \phi^p d\mu \right)^{\frac{2-p}{p}} \left[ \frac{1}{(n\omega_n)^{\frac{1}{p-1}}} \int_{D^*} \phi^p d\mu \right]^{\frac{p-2}{p}} \\
&= (n\omega_n)^{\frac{2-p}{p}} \mathcal{C}_p(D^*). \quad \square
\end{aligned}$$

In order to obtain a lower bound for  $\Lambda_*$  in terms of  $\mathcal{C}_p(D^*)$ , we first need to relate the particular  $\Lambda$  occurring in the Euler-Lagrange equation (2.18) to the eigenvalue  $\Lambda_*$ , just as (1.3) relates the number  $\lambda$  occurring in the Euler-Lagrange equation (1.2) to the eigenvalue  $\mathcal{C}_p(D)$ .

**Lemma 9.** *Let  $f$  be a minimizer for  $\Lambda_*$  given by (2.16) with the boundary conditions (2.17) and satisfy the Euler-Lagrange equation (2.18), written as*

$$\frac{d}{d\rho} \left[ \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{1}{p-1}} \right] + \Lambda \rho^{1-n} f(\rho) = 0. \quad (2.20)$$

Then

$$\Lambda = \Lambda_* \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \right)^{\frac{2-p}{p}}. \quad (2.21)$$

*Proof.* Multiply the Euler-Lagrange equation (2.20) across by  $f(\rho)$  and integrate from 0 to  $\rho_M$  to obtain

$$\int_0^{\rho_M} f(\rho) \frac{d}{d\rho} \left[ \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{1}{p-1}} \right] d\rho + \Lambda \int_0^{\rho_M} \rho^{1-n} f(\rho)^2 d\rho = 0.$$

Integrating by parts in the first term and using the boundary conditions (2.17) gives

$$\int_0^{\rho_M} f(\rho) \frac{d}{d\rho} \left[ \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{1}{p-1}} \right] d\rho = - \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho,$$

from which it follows that

$$\Lambda = \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \bigg/ \int_0^{\rho_M} \rho^{1-n} f(\rho)^2 d\rho.$$

We can use (2.16) to write  $\int_0^{\rho_M} \rho^{1-n} f^2(\rho) d\rho$  in terms of  $\Lambda_*$  since  $f$  is a minimizer for this Rayleigh quotient, leading to

$$\Lambda = \Lambda_* \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \right)^{1 - \frac{2(p-1)}{p}},$$

which is (2.21). □

**Lemma 10.**

$$\mathcal{C}_p(D^*) \leq (n\omega_n)^{\frac{p-2}{p}} \Lambda_*. \quad (2.22)$$

*Proof.* Let  $f$  be a minimizer for the generalized quotient (2.16) defining  $\Lambda_*$ . Set

$$\psi(\rho) = \int_{\rho}^{\rho_M} r^{1-n} f(r) dr, \quad 0 \leq \rho \leq \rho_M,$$

so that  $\psi(\rho_M) = 0$ . Then  $\psi(\rho)$  (where  $\rho = |x|$  for  $x \in \mathcal{C}_p(D^*)$ ) is an admissible test function for the quotient defining  $\mathcal{C}_p(D^*)$ . Thus

$$\mathcal{C}_p(D^*) \leq (n\omega_n)^{\frac{p-2}{p}} \int_0^{\rho_M} \rho^{n-1} \psi'(\rho)^2 d\rho \Big/ \left( \int_0^{\rho_M} \rho^{n-1} \psi(\rho)^p d\rho \right)^{2/p}. \quad (2.23)$$

Now

$$\int_0^{\rho_M} \rho^{n-1} \psi'(\rho)^2 d\rho = \int_0^{\rho_M} \rho^{n-1} [\rho^{1-n} f(\rho)]^2 d\rho = \int_0^{\rho_M} \rho^{1-n} f(\rho)^2 d\rho. \quad (2.24)$$

Next, using the Euler-Lagrange equation (2.20),

$$\begin{aligned} \psi(\rho) &= \int_{\rho}^{\rho_M} r^{1-n} f(r) dr \\ &= -\frac{1}{\Lambda} \int_{\rho}^{\rho_M} \frac{d}{dr} \left[ r^{\frac{1-n}{p-1}} f'(r)^{\frac{1}{p-1}} \right] dr \\ &= -\frac{1}{\Lambda} \left. r^{\frac{1-n}{p-1}} f'(r)^{\frac{1}{p-1}} \right|_{r=\rho}^{r=\rho_M} \\ &= \frac{1}{\Lambda} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{1}{p-1}}, \end{aligned}$$

where we used  $f'(\rho_M) = 0$ . From this we obtain that

$$\begin{aligned} \int_0^{\rho_M} \rho^{n-1} \psi(\rho)^p d\rho &= \int_0^{\rho_M} \rho^{n-1} \frac{1}{\Lambda^p} \rho^{\frac{p(1-n)}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \\ &= \frac{1}{\Lambda^p} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho. \end{aligned} \quad (2.25)$$

With the help of the identities (2.24) and (2.25), we can write the numerator and the denominator of (2.23) in terms of the minimizer  $f$  for  $\Lambda_*$ . We find, using that  $f$  minimizes the quotient for  $\Lambda_*$  at the second step and using (2.21) at the third step, that

$$\mathcal{C}_p(D^*) \leq (n\omega_n)^{\frac{p-2}{p}} \int_0^{\rho_M} \rho^{1-n} f(\rho)^2 d\rho \Big/ \left( \frac{1}{\Lambda_*^p} \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \right)^{\frac{2}{p}} \quad (2.26)$$

$$= (n\omega_n)^{\frac{p-2}{p}} \frac{\Lambda_*^2}{\Lambda_*} \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \right)^{\frac{2(p-1)}{p} - \frac{2}{p}} \quad (2.27)$$

$$= (n\omega_n)^{\frac{p-2}{p}} \frac{1}{\Lambda_*} \left[ \Lambda_* \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} f'(\rho)^{\frac{p}{p-1}} d\rho \right)^{\frac{p-2}{p}} \right]^2 \quad (2.28)$$

$$= (n\omega_n)^{\frac{p-2}{p}} \frac{\Lambda_*^2}{\Lambda_*} = (n\omega_n)^{\frac{p-2}{p}} \Lambda_*.$$

□

## 2.4 Completion of the proof of Theorem 1

We can now finally complete the proof of Theorem 1. Indeed, we have

$$\begin{aligned}
\int_0^{\rho_M} \rho^{1-n} H^2(\rho) d\rho &\leq \frac{1}{\Lambda_*} \left( \int_0^{\rho_M} \rho^{\frac{1-n}{p-1}} \left( \frac{dH}{d\rho} \right)^{\frac{p}{p-1}} d\rho \right)^{\frac{2(p-1)}{p}} \\
&= \frac{1}{\Lambda_*} \left( (n\omega_n)^{\frac{1}{p-1}} \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \\
&= \frac{1}{\Lambda_*} (n\omega_n)^{2/p} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}},
\end{aligned}$$

with equality if and only if  $D$  is a ball and  $\phi$  is radial. Substituting this last inequality into (2.10), we have

$$\begin{aligned}
\left( \int_D \phi^{p-1} d\mu \right)^2 &\leq \frac{2n^2 \omega_n^{n/2}}{p \mathcal{C}_p(D)} |D|^{\frac{n-2}{n}} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}} \\
&\quad - (n-2) \omega_n^{\frac{2-n}{n}} |D|^{\frac{n-2}{n}} \frac{1}{\Lambda_*} (n\omega_n)^{2/p} \left( \int_D \phi^p d\mu \right)^{\frac{2(p-1)}{p}}.
\end{aligned}$$

Since  $\Lambda_* = (n\omega_n)^{\frac{2-p}{p}} \mathcal{C}_p(D^*)$  by (2.19) and (2.22), the main inequality (1.4) follows with equality if and only if  $D$  is a ball.  $\square$

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